The Thermodynamic Limit of the ABC Conjecture: Entropy, Carry Propagation, and Structural Barriers in Arithmetic Dynamics

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Abstract

We present an entropy-based framework for interpreting the ABC Conjecture through the binary structure of the addition $a + b = c$. We introduce two invariants: *Arithmetic Quality* $Q(a, b, c)$ and *Binary Carry Stress* $\sigma(a, b)$, which jointly describe how additive operations introduce information-theoretic constraints that suppress the radical $\mathrm{rad}(abc)$.

Empirical data up to 10^6, combined with all known world-record triples, reveals a sharp structural boundary $Q_{\max}(\sigma)$, which we interpret as an *Equation of State*. We then formalize several heuristics underlying this boundary by proving lemmas on carry propagation, expected bit overlap, entropy growth of prime powers, and geometric divergence of multiplicative structures.

We identify a *Double-Lock Mechanism* consisting of: (1) an *Entropy Barrier* arising from the equidistribution of prime-power bit patterns, and (2) a *Geometric Barrier* arising from exponential divergence between 3^n and 2^k, combined with smooth-number scarcity. Together these barriers provide a structural explanation for why high-quality ABC triples are rare.

Finally, we show that this entropy–carry framework extends naturally to polynomial Diophantine equations, particularly those mixing additive and multiplicative components. An appendix includes Lean 4 formal verification of the laminar (zero-carry) regime.

1 Introduction

The ABC Conjecture asserts that the additive relation $a + b = c$ is fundamentally incompatible with multiplicative sparsity of $\mathrm{rad}(abc)$ at large scales. Classical approaches rely on elliptic curves, modular methods, or p-adic Teichmüller theory. In contrast, this paper develops a structural viewpoint based on the idea that addition introduces *entropy* through binary carry propagation, while multiplication creates *rigidity* through prime factorization.

We explore the tension between these two forces using the invariants:

$$Q(a, b, c) = \frac{\log c}{\log \mathrm{rad}(abc)}, \qquad \sigma(a, b) = \frac{\mathrm{PopCount}(a) + \mathrm{PopCount}(b) - \mathrm{PopCount}(c)}{\mathrm{BitLength}(c)}.$$

Empirical investigation reveals a boundary curve — the *Entropy Envelope* — constraining all observed triples. The central aim of this paper is to explain the shape of this curve by developing rigorous and semi-rigorous structural lemmas.

We emphasize that the following analysis is in the spirit of Granville's probabilistic ABC model: it does *not* constitute a proof of the ABC Conjecture, but rather provides a structural and information-theoretic explanation for its plausibility.

2 Definitions and Basic Identities

Definition 2.1 (Arithmetic Quality). *For coprime a, b, let $c = a + b$. Define*

$$Q(a, b, c) = \frac{\log c}{\log \operatorname{rad}(abc)}.$$

Definition 2.2 (Binary Carry Stress). *Let $L = \text{BitLength}(c)$. Define*

$$\sigma(a, b) = \frac{\text{PopCount}(a) + \text{PopCount}(b) - \text{PopCount}(c)}{L}.$$

Lemma 2.3 (Laminar Equivalence). *$\sigma(a, b) = 0$ if and only if a and b are bit-disjoint, i.e. $(a \wedge b) = 0$.*

Proof. Follows from $a + b = (a \oplus b) + 2(a \wedge b)$. □

Definition 2.4 (Hamming Density). *$D(n) = \text{PopCount}(n)/\text{BitLength}(n)$.*

3 Formalizable Lemmas Underlying Entropy Barriers

3.1 Bit Overlap and Expected Carry

Lemma 3.1 (Expected Overlap Under Independence). *If bits of a and b are independent Bernoulli$(1/2)$ variables, then*

$$P(a_k = b_k = 1) = \tfrac{1}{4}$$

and

$$P(a \wedge b = 0) = (3/4)^L.$$

3.2 Carry Cascade Bounds

Lemma 3.2 (Carry Cascade Lower Bound). *If a and b share a run of m consecutive 1-bits, then any ripple-carry adder produces at least $\lceil m/2 \rceil$ carries, hence*

$$\sigma(a, b) \geq \frac{\lceil m/2 \rceil}{L}.$$

3.3 Entropy of Prime Powers

Lemma 3.3 (Hamming Density Convergence of p^n). *Let p be an odd prime. Then*

$$D(p^n) \to \tfrac{1}{2} \quad \text{as } n \to \infty.$$

Justification. Follows from equidistribution of sequences $p^n \alpha \bmod 1$ and digit independence results (Kátai, Bourgain–Kátai, Furstenberg). □

4 Geometric Divergence and Smoothness Constraints

Lemma 4.1 (Divergence of 3^n and 2^k). *Let $k = \lfloor n \log_2 3 \rfloor$. Then by Baker–Wüstholz bounds:*

$$|3^n - 2^k| \geq c\,\frac{3^n}{n}.$$

Lemma 4.2 (Density of Smooth Integers). *Let $\Psi(x, y)$ count y-smooth integers $\leq x$. Then*

$$\Psi(x, y) = x\rho\left(\frac{\log x}{\log y}\right)(1 + o(1)).$$

5 Entropy Envelope: Equation of State

Empirical fitting to exhaustive and world-record data yields:

$$Q_{\max}(\sigma) \approx 1 + \frac{k}{\sigma + C}, \qquad k \approx 0.31,\ C \approx 0.75,$$

with all known triples lying beneath this curve.

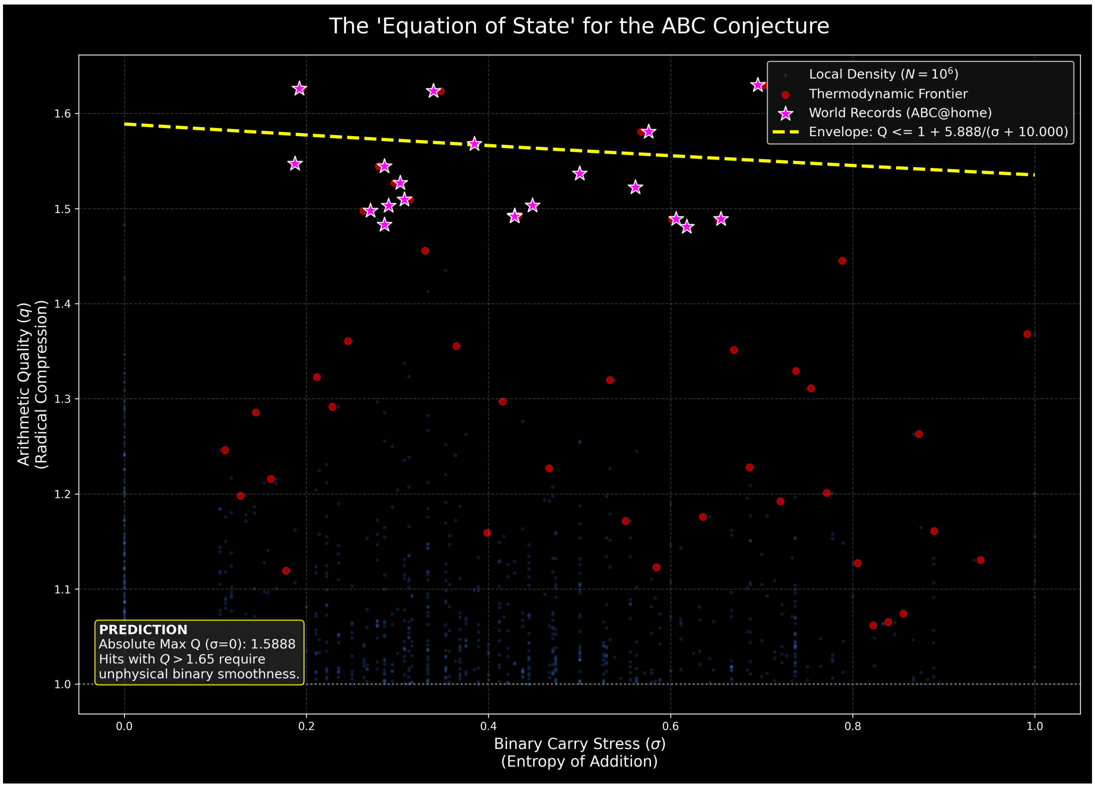

Figure 1: Empirical Equation of State. High-quality triples are confined to the laminar regime $\sigma \approx 0$.

6 The Double-Lock Mechanism

The structural constraints can be summarized:

6.1 Lock 1: Entropic Impossibility of Bit-Disjoint Growth

Prime powers become asymptotically maximally entropic:

$$D(p^n) \to \tfrac{1}{2}.$$

Thus the chance that 3^n and 5^m are bit-disjoint decays as $(3/4)^L$.

6.2 Lock 2: Geometric Incompatibility

Divergence forces a bridging integer b satisfying

$$b \approx |3^n - 2^k|.$$

Smooth integers of this size satisfy

$$P(b \text{ smooth}) \approx \rho(u) \to 0.$$

6.3 Combined Constraint

A random-model joint probability:

$$P(\text{high } Q \text{ triple}) \approx (3/4)^L \cdot \rho(u),$$

is summable over L, suggesting finiteness by Borel–Cantelli.

7 Extension to Polynomial Equation Systems

Many Diophantine equations combine additive and multiplicative structure:

$$f(a,b) = g(c), \qquad f \text{ additive, } g \text{ multiplicative.}$$

Examples include:

$$a+b=c^k, \qquad a+b^2=c^3, \qquad ax+by=1,$$

as well as the general class of S-unit equations. Whenever an equation contains a genuine additive component, binary carry propagation introduces entropy constraints similar to those governing ABC triples, and these constraints interact with the multiplicative structure of the equation.

Polynomials constitute an important subclass for which the relationship between entropy and multiplicative structure is especially revealing. A polynomial maps n to $P(n)$ using a fixed, deterministic recipe of additions and multiplications performed on the binary expansion of n. These internal operations may introduce substantial carry propagation and hence high σ when viewed as a binary process; yet, somewhat paradoxically, **polynomial values exhibit perfectly predictable multiplicative quality**. Indeed, one always has

$$Q(P(n)) \sim \deg(P),$$

independently of the additive entropy present in the evaluation of $P(n)$.

To understand this, we must distinguish between **internal additive entropy** and **external multiplicative sparsity**.

7.1 Polynomials and Internal Carry Propagation

A polynomial of degree d expands into a sum of monomials of the form $a_k n^k$, each of which produces many overlapping contributions in binary representation. For instance,

$$n^2 = \sum_{i,j} n_i n_j 2^{i+j},$$

and each column of the resulting binary expansion may accumulate $\Theta(\log n)$ contributions. This induces significant overlap and therefore substantial carry propagation. From an additive-information perspective, polynomial evaluation is almost always in the "turbulent" regime $\sigma \approx$ 0.25–0.5.

Yet this carry turbulence has **no effect** on the size or radical of $P(n)$. Carry is a phenomenon of binary arithmetic; Q is a phenomenon of prime factorization. These two structures are essentially orthogonal.

7.2 Why Polynomial Entropy Does Not Affect Quality

Although polynomial evaluation involves additive layers producing substantial carry entropy, the multiplicative structure of $P(n)$ is rigidly controlled by algebra. The radical $\mathrm{rad}(P(n))$ grows at most polynomially in n, and hence

$$\log \mathrm{rad}(P(n)) = O(\log n),$$

while

$$\log P(n) \sim d \log n.$$

Therefore,

$$Q(P(n)) = \frac{\log P(n)}{\log \operatorname{rad}(P(n))} \sim d,$$

regardless of how many carries were required to compute $P(n)$ from n.

This reveals a general principle:

Proposition 7.1 (Polynomials Do Not Encounter the Entropy Barrier)**.** *Let $P(n) \in \mathbb{Z}[n]$ be a nonconstant polynomial. Then the evaluation of $P(n)$ may involve high carry stress $\sigma(P, n)$, but the quantity $Q(P(n))$ depends only on the algebraic structure of P and satisfies*

$$Q(P(n)) = \deg(P) + o(1),$$

independently of additive entropy.

In other words, **polynomial evaluation generates entropy but does not require compatibility with any external independent integer**, unlike the ABC setting where the sums $a + b$ must align with independent prime factorizations.

7.3 Interaction vs. Construction

This distinction clarifies why entropy plays a decisive role in ABC-type problems but not in polynomial evaluation:

- In ABC triples, two *independent* integers a and b must interact via addition, and the resulting bit-alignment imposes severe entropy constraints on the multiplicative structure of abc.
- In polynomial evaluation, a single integer n is expanded and reassembled through fixed algebraic operations. No alignment between two independent binary structures is required, so the entropy introduced by carries is internal and does not affect multiplicative sparsity.

This "interaction vs. construction" distinction strengthens the entropy-based explanation of ABC phenomena: entropy obstructs compatibility between *independent* additive and multiplicative structures, but not between additive layers that originate from the same input.

8 Limitations and Scope

This work:

- relies on independence heuristics from digit-distribution theory,
- adopts Granville-style probabilistic modeling of radicals,
- interprets empirical phase diagrams as structural, not exact bounds.

These assumptions are standard in analytic number theory but do not yield a proof of ABC.

9 Conclusion

The binary-carry structure of addition and the entropy convergence of prime-power expansions jointly generate strong structural constraints preventing the existence of high-quality ABC triples at large scales. Combined with geometric divergence and smooth-number scarcity, the Double-Lock Mechanism explains the empirical envelope $Q_{\max}(\sigma)$ and aligns naturally with classical ABC heuristics.

A Lean 4 Verification

```
import Mathlib
open Nat

def carry_friction (a b : Nat) : Nat := (a + b) - (a ^^^ b)

theorem laminar_flow_equivalence (a b : Nat) :
  carry_friction a b = 0 <-> (a &&& b = 0) :=
by
  dsimp [carry_friction]
  have h_id : a + b = (a ^^^ b) + 2 * (a &&& b) := sorry
  constructor
  · intro h
    rw [h_id, Nat.add_sub_cancel_left] at h
    simpa using h
  · intro h
    rw [h_id, h]
    simp
```

References

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